Extended Abstract

This thesis is, first and foremost, about 3-dimensional oriented TQFTs. These have an already long history within the field of low-dimensional topology, and recent applications to condensed matter physics have renewed the demand for them to be carefully examined. The general ethos, when it comes to 3d-TQFTs is given by:

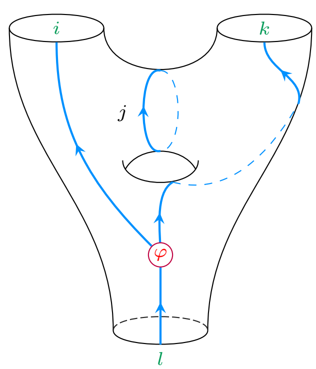
$$\text{3-dimensional TQFTs} \longleftrightarrow \text{Modular Tensor Categories} \tag{0.1}$$

A lot of the seminal work done by Kirillov Jr. and Bakalov [2] on this subject, revolves around trying to establish 0.1. Although this is a nice guiding principle, and a fruitful research slogan, it underplays a rich story. For one, we care about TQFTs that extend beyond Atiyah's program. In particular, we are interested in developing theories that have greater *locality*. By locality we mean the notion, inspired by physicists rejection of any spooky actions at a distance, that one should be able to tell what TQFT one has just by understanding its behaviour on the neighborhood of a point in spacetime, or, ideally, to uderstanding its behaviour at every point. In other words, we are interested in a class of TQFTs called *fully extended*, whose classification was proposed by Baez and Dolan under the framework of the *Cobordism Hypothesis* [1], and for which a proof has been sketeched by Lurie and Hopkins [13]. In the case of oriented TQFTs, the cobordism hypothesis establishes a 1-1 mapping between fully extended 3-dimensional TQFTs and $\mathrm{SO}(3)$-homotopy fixed-points in the 3-category of $\mathbb{C}$-linear monoidal categories — which are conjectured [6] to be *Spherical Fusion Categories* [4].

Complementing this result, in a recent paper by Bartlett-Douglas-Schommer–Pries-Vicary [5], it is proved that oriented 3d-TQFTs that are once-extended i.e. extended down to the circle, are in 1-1 correspondance with anomaly-free modular tensor categories (these are also called theories with trivial central-charge in Conformal Field Theory), examples of which are given by taking the center $\mathcal{Z}(\mathcal{C})$ of a spherical tensor category $\mathcal{C}$ (see Remark 5.19 in [14]).

In this thesis we consider how one can take the hint from the cobordism hypthesis, by focusing on building oriented TQFTs from a spherical fusion category $\mathcal{C}$, which we will call String-Net TQFT, and showing that they correspond, when interpreted as only once-extended, to the MTCs given by taking the Drinfel'd center of $\mathcal{C}$. In the process we show that, as the cobordism hypothesis suggests, the String-Net construction is equivalent to the other famous TQFT defined using a spherical fusion category, the Turaev-Viro-Barret-Westbury state-sum [15][3], TVBW for short. The advantage here is that String-Nets are far simpler to build and interpret than the state-sum. We also try and shed light on the conjecture that fully extended TQTFs are in 1-1 correspondence with Topological Phases of Matter by proving that the Levin-Wen model, believed to capture the universal properties for these phases, is a Hamiltonian realization of the String-Net Fiel Theory.

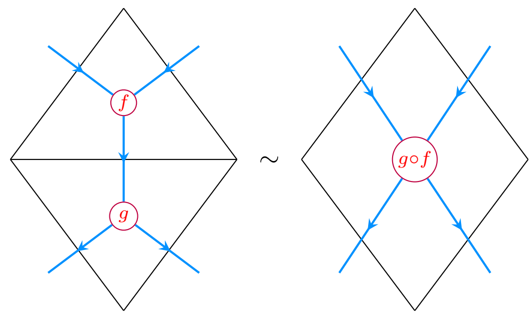
The String-Net construction, as a TQFT, first appeared in Walker's work [16] as an example of a broader construction called TQFTs via *topological fields and local relations*. Put bluntly, as a 3-dimensional TFQT (let us call it $\mathcal{Z}$, for now), it assigns to every oriented surface Σ (without corners), a Hilbert space constructed out of all $\mathbb{C}$-linear combinations of graphs properly embedded in Σ, decorated with the data of some spherical fusion category $\mathcal{C}$. These decorated, or labelled graphs, are what Walker calls topological fields, an example of what we mean by a decorated graph is given in the picture below.



(0.2)

As we will see, the fact that we are making use of a spherical category, in particular a pivotal category, allows us to interpret the pictures inside any embedded disk as a string-diagram in $\mathcal{C}$. By converting an embedded disk, call it

$D^2 \subset \Sigma$, to an equivalent string-diagram i.e. one corresponding to the same morphism in $\mathcal{C}$, we construct what are called *local relations* and we dub this operation *evaluation* in D^2. Local relations correspond to the set of equations between any pair of decorated graphs that differ only by a finite sequence of applications of evaluation map. The Hilbert space associated to Σ, is the space of all $\mathbb{C}$-linear decorated graphs in Σ, modded out by the local relations prescribed above. The fact that this vector space is a topological invariant of the surface (in the case the surface is closed) has to do with how the local relations provided by a spherical fusion category mimic the typical subdivision and mutation moves that are associated to cellular-decompositions of surfaces. A stand-out simple example to get to grips with this way of relating labelled graphs is the relation between removing an edge of the cellular-decomposition and the composition of 1-morphisms in $\mathcal{C}$:

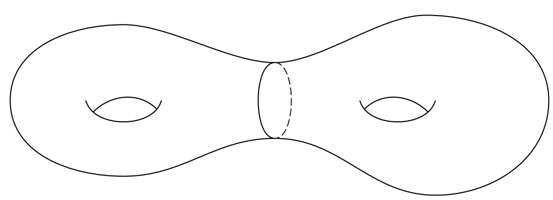


We would say that the left string-diagram is evaluated to the same string-diagram on the right. We sum up the construction for closed manifolds as follows:

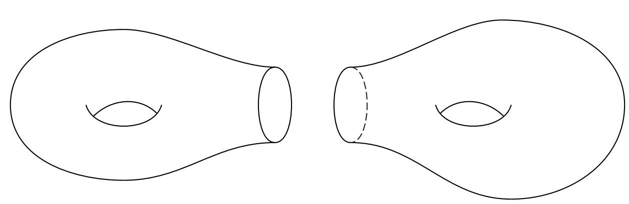
$$\mathcal{Z}(\Sigma) \;:=\; \mathbb{C}\{\text{decorated graphs in } \Sigma\}/\{\text{local relations}\} \tag{0.3}$$

This naturally extends to manifolds without corners, as is suggested by the picture 0.2, so long as we refrain from applying any local relation carried out by an evaluation intersecting the boundary circles of the surface. This suggests that, for any given oriented surface with non-empty boundary and with an embedded labelled graph Γ there is a boundary condition that records how the labelled graph intersects the boundary. In other words, boundary conditions amount to oriented circles $B_1, \ldots, B_k$ with marked points labelled by objects of $\mathcal{C}$, which we denote as $\{V_{b_1}\}_{b_1 \in B_1}, \ldots, \{V_{b_k}\}_{b_k \in B_k}$ — we will often use short-handed versions of this notation of boundary conditions.

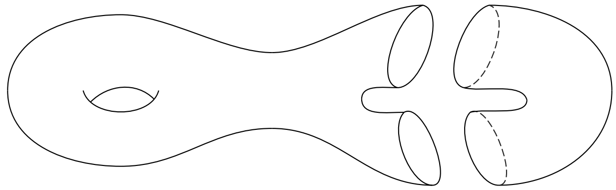
The next step is to use the fact that, according to the cobordism hypothesis, the string-net TQFT is fully-extended and in particular once-extended. This means that the vector space $\mathcal{Z}(\Sigma)$ has to be able to be computed by chopping Σ as the gluing of simpler surfaces (most likely with boundary) and then apply some gluing rules using the vector space associated to each of the simpler pieces. If Σ is the 2-holed torus



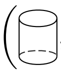
then we can write Σ as the gluing of the following two once-punctured toruses

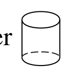


or, alternatively, we can write it as the gluing of a cylinder with a twice-punctured torus



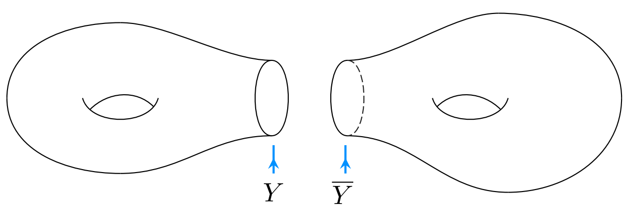
the combinations for additional splittings and regluings are quite literally endless. We have mentioned gluing rules — when our objects of study are the standard non-extended parts of a TQFT, the gluing of two bordisms is algebraically carried out by composing linear maps — and understanding them for higher-codimensional splittings involve understanding the 1-categorical data that is encoded in the boundary circles of the oriented surfaces we have been examining. That 1-categorical data is what we will call the annularization of $\mathcal{C}$ or the tube category associated to $\mathcal{C}$ — which we will call $\mathrm{Tube}_{\mathcal{C}}(S^1)$. Its objects are given as boundary conditions $\{V_b\}_{b \in B}$ with the morphisms given by

$\mathrm{Hom}(\{V_b\}, \{W_c\}) \;:=\; \mathcal{Z}\Bigg($  $, \{V_b\}^\vee, \{W_c\}\Bigg)$

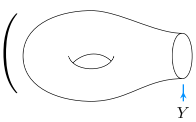
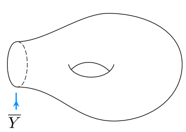
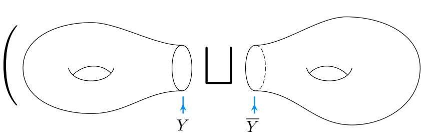
where the $\vee$ will be explained later — it is related to the fact the we consider  with an incoming and an outgoing circle and composition is given by gluing (stacking tubes). And we define

$$\mathcal{Z}(S^1) := \mathrm{Tube}_{\mathcal{C}}(S^1) \tag{0.4}$$

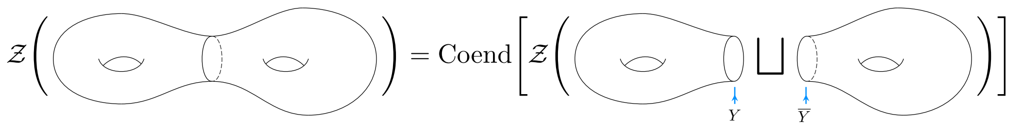
As we will explore in greater detail, by fixing a set of boundary conditions, the vector spaces associated to these surfaces afford modules for the tube category, where the action is defined by gluing string-net-colored collars to the boundary circles. Picking up the example of the two-holed torus again



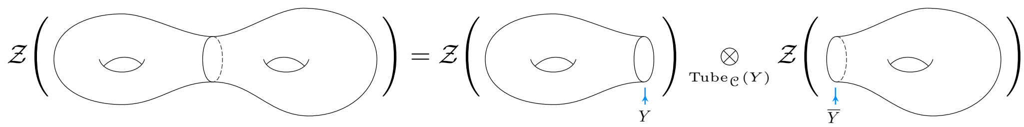
then we have that:

- $\mathcal{Z}\Bigg($  $\Bigg)$ is a $\mathrm{Tube}_{\mathcal{C}}(Y)$-module;
- $\mathcal{Z}\Bigg($  $,\Bigg)$ is a $\mathrm{Tube}_{\mathcal{C}}(-Y) = \mathrm{Tube}_{\mathcal{C}}(Y)^{\mathrm{op}}$-module;
- $\mathcal{Z}\Bigg($  $\Bigg)$ a $\mathrm{Tube}_{\mathcal{C}}(Y) \times \mathrm{Tube}_{\mathcal{C}}(Y)^{\mathrm{op}}$-module.

what we manage then to prove is that:

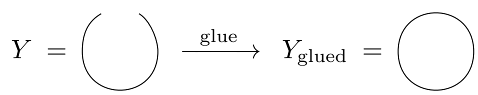


or as the coend is more often referred to, when applied to the language of representations:



it corresponds to the tensor product of a left and right module of $\mathrm{Tube}_{\mathcal{C}}(Y)$. This provides the gluing rule (or schema) for gluing along codimension-1 manifolds without corners. This is an extremely useful result as we will see, because it introduces a formula that we can leverage to understand how data, encoded in the vector space of a surface, might be localized to a particular region. Moreover, it introduces a greater algebraic understanding to the cut and paste nature of string-diagrams and is backed by a vast literature on (co)ends.

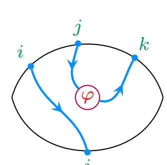
This construction can be generalized in different ways. The most extensive, and laborious one to achieve, is to keep going down the categorical loop-hole and to attribute a 2-category to the point. In our construction this 2-category is the bicategory $\mathcal{C}$ with a unique object, meaning that the point is always attributed the same algebraic structure i.e. $\mathcal{C}$. This is not the laborious aspect of the construction, that would be understanding how to glue 1-manifolds along their corners. For example:



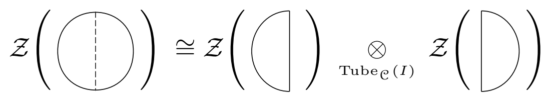
For the example above, any gluing formula would involve a general understanding of birepresentation theory. This is a consequence of the fact that, as opposed to what happens with surfaces, where the algebraic objects are vector spaces here the glued manifold is really a 1-manifold, implying that our categorified gluing needs to recapture the 1-category associated to the circle as a sort of categorified coend, 2-coend or 2-tensor product over the bicategory associated to the point. The other generalization route is to consider surfaces with corners — the reason why this pursuit ends up being not as complicated as the one mentioned above, is twofold: first, we are still just gluing surfaces along 1-manifolds, and hence we dispense the need for higher categorical tools, moreover since our 2-category is really just the delooping of a 1-category, our corners are handled rather trivially.

We defined the tube category as the category whose objects are boundary conditions to string-nets of surfaces with boundary i.e. circles with a decorated 1-graph. To comprehend surfaces with corners, we must extend the tube category to all 1-manifolds, therefore, for open intervals we consider objects to be decorated 1-graphs without vertices at the endings:

where the morphisms are given by string-nets on the bigon (also called pinched interval cobordism of the intreval) whose boundary is split by the source and target objects, respectively. An example of a morphism in this generalized tube category is given by the following diagram:



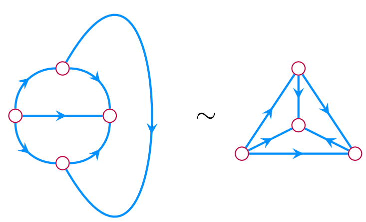
Just like the string-net vector spaces, for manifolds without corners, provide a module for the S^1 tube category, the string-net vector spaces for surfaces with corners provide modules for the interval tube category. And through the application of an argument *mutatis mutandis* to the cornerless case, we get that:


(0.5)

which offers an extension of our gluing/cut-and-paste toolkit.

Before trying to relate the string-net construction to anything else, it is useful to add a remark on the nature of spherical fusion categories. In other words, provide reasoning on why they seem to be linked to topological quantum field theories in the first place. Ever since the beginning of quantum topology, between the mid-1980's and early 1990's, particular attention was cast on the role of *Quantum Groups* as a useful algebraic gadget to construct invariants of links — now aptly named quantum invariants, whose distinguishing power extended beyond the classical tools of *knot theory* i.e. homotopy groups of the knot's complement — and as a consequence produce 3-manifold invariants subject to being constructed via a 3d-TQFT. Quantum groups, opposite to what their name might suggest, are really algebras, and in particular Hopf algebras with some additional structure and whose representation categories have the structure of spherical tensor categories.

With this historical background past us, it is sensible to state, in rough terms, why spherical fusion categories, as an intrinsically algebraic object, can be manipulated as pictures or string-diagrams. We begin by recalling that for tensor categories there is an extremely useful diagrammatic language, defined by representing objects as 1-dimensional strands and morphisms as boxes, where composition is given by contracting strands and tensoring by horizontal juxtposing whatever is being tensored. This allows one to evaluate complicated algebraic equations by manipulating these strands in topologically "sensible" ways. Spherical categories gather all the necessary algebraic data to ensure full isotopy-invariance of the aforementioned diagrammatic calculus and guarantees that any closed graph can be interpreted as an endomorphism of the monidal unit $\mathrm{End}(\mathbb{1}, \mathbb{1}) \cong \mathbb{C}$ (which is an axiom, not a theorem). As such, and as an example, the following pair of graphs (we omit the labelling)



are evaluated to same morphism in $\mathrm{End}(\mathbb{1}, \mathbb{1}) \cong \mathbb{C}$ i.e. the same scalar. Hopefully, this provides a clearer picture of why these algebraic constructions are useful when probing topological properties. One of the most important properties of these categories is that any object can be written as a direct sum of what are called simple objects $X_i \in \mathrm{Irr}(\mathcal{C})$. Moreover, the list of simple objects is assumed to be finite and the morphism spaces between them must satisfy:

$$\mathrm{Hom}(X_i, X_j) \cong \delta_{i,j}\, \mathbb{C} \tag{0.6}$$

We call these semisimple categories and often refer to them as fusion categories because of the way they have been used by physicists to study fusions of particles, defects and domain walls in certain QFTs, giving rise to the name fusion algebra for the set of coefficients N_{ij}^k appearing in the decomposition into simple objects:

$$X_i \otimes X_j \cong \bigoplus_{X_k \in \mathrm{Irr}(\mathcal{C})} N_{ij}^k \cdot X_k \tag{0.7}$$

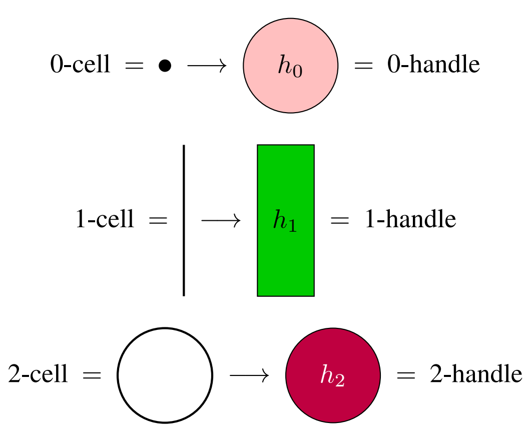
where $N_{ij}^k \cdot X_k := X_k \oplus \cdots \oplus X_k$ has N_{ij}^k components. The interplay between these fusion rules and the rest of the categorical data determine much of what we had previously called local relations. They are equations, typically written in the pictorial language we have been describing, and encapsulate structural data of the category. Examples of these categories come in varied flavours:

- Vect_k — the category of finite-dimensional vector spaces of an algebraically closed field k;
- $\mathrm{Vect}_{\mathbb{C}}[G]$ — the category of G-graded vector spaces over the complex numbers. where G is some finite group;
- $\mathrm{Rep}(G)$ — the category of finite-dimensional representations of G over the complex numbers, for some finite group G;
- $\mathrm{Rep}(U_q(\mathfrak{g}))$ — the (sub)category of (appropriately reduced to quotients by tilting modules) of representations of the universal enveloping algebra of a semisimple Lie algebra $\mathfrak{g}$, and such that q is a root of unity.

The list goes on and has many exotic examples which are still poorly understood, especially for categories with numbers of simple objects above 5, one of the great challenges being the absence of a counterpart to simple groups in finite group classification. The work on subfactors here has proved extremely useful.

Following this prelude to the first chapter on spherical fusion categories, we skip to a soft, albeit general discussion, of how to take the string-net construction to setup a sum of locally commuting projectors (a Hamiltonian), for any given oriented closed surface Σ, and for which the ground-state is canonically isomorphic to $\mathcal{Z}(\Sigma)$. This should serve as a stepping stone to the discussion of the Levin-Wen model later in the thesis, but it is meant, first and foremost, to bridge the gap between local or fully extended TQFTs and the study of physical models with emergent "topological" properties, also called topological phases of matter.

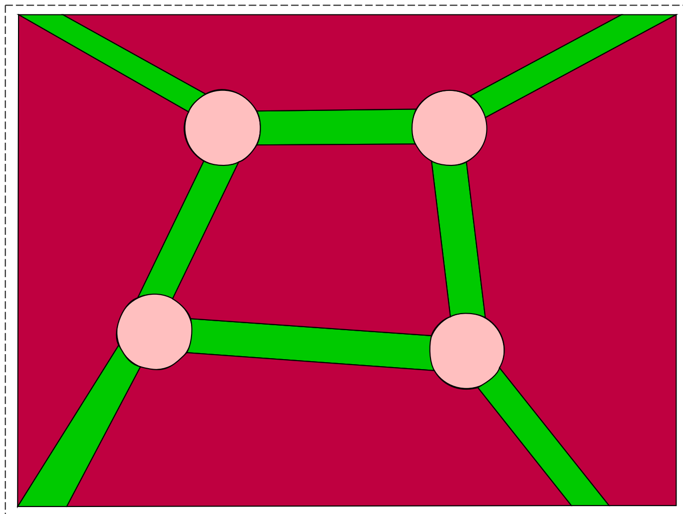
Let Σ be an oriented closed surface and let H_Σ be a handle decomposition of Σ. For the unfamiliar with the last sentence, consider instead a generic cellular decomposition of Σ and "fatten" every k-skeleton in this decomposition. For a 2-dimensional handle decomposition we are restricted to 0-cells, 1-cells and 2-cells or 0-handles, 1-handles and 2-handles. The correspondence between the two is given as follows:



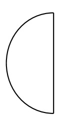
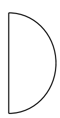
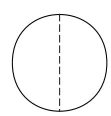
More formally, a 2-dimensional k-handle is identified with a manifold homeomorphic to $h_k = D^k \times D^{2-k}$. We split its boundary

$$\partial(h_k) = \partial(D^k \times D^{2-k}) = (\partial(D^k) \times D^{2-k}) \cup (D^k \times \partial(D^{2-k})) = (S^{k-1} \times D^{2-k}) \cup (D^k \times S^{2-k-1}) \qquad (0.8)$$

into what we call an attaching boundary (the set on the left-hand side of the boundary decomposition) and a non-attaching boundary (the set on the right-hand side of the boundary decomposition. The following picture is chosen to exemplify a local patch of a generic 2-dimensional handle decomposition.



Taking this handle decomposition H_Σ how can we construct the commuting projectors we are looking for? Our aim is to build a set of projectors whose images, when intersected, give us a space isomorphic to the space of string-nets in Σ. One way to do this is achieved by taking a closer look at the gluing formulas we presented in the beginning of the introduction. Let us rewrite equation 0.5, but first establish some names:

$\Sigma_1 :=$  $\bigsqcup$ $\Sigma_2 :=$  $\xrightarrow{\text{glue}} \Sigma_1 \cup_I \Sigma_2 =$  (0.9)

and we define I to be the interval component of the boundary along which we will glue both disks. Then equation 0.5 can be rewritten as

$$\mathcal{Z}(\Sigma_1 \cup_I \Sigma_2) \cong \left[\bigoplus_{\{V_b\} \in \mathrm{Tube}_{\mathcal{C}}(I)} \mathcal{Z}(\Sigma_1, \{V_b\}) \otimes \mathcal{Z}(\Sigma_2, \{V_b\}) \right] \Big/ \langle (\Gamma_1 \cdot \alpha) \otimes \Gamma_2 \sim \Gamma_1 \otimes (\alpha \cdot \Gamma_2) \rangle \qquad (0.10)$$

where Γ_1 and Γ_2 are string-net configurations in both Σ_1 and Σ_2, respectively, and α is a string-net configuration of the bigon i.e. a morphism of $\mathrm{Tube}_{\mathcal{C}}(I)$ and the $\cdot$ is notation for the module action. Since both $\mathrm{Tube}_{\mathcal{C}}(S^1)$ and $\mathrm{Tube}_{\mathcal{C}}(I)$ are semisimple categories (see [14]), there must exist a set of finite idempotents, let us call them

$$\alpha_i \colon \{V_{b_i}\}_{b_i \in B_i} \longrightarrow \{V_{b_i}\}_{b_i \in B_i},$$

which one can use to decompose any module M into simple submodules. In other words, given the action $\{\pi_\alpha \colon M \longrightarrow M\}_{\alpha \in \mathrm{Tube}_{\mathcal{C}}(I)}$, we can decompose any module M as

$$M \cong \bigoplus_{\alpha_i} \mathrm{im}(\pi_{\alpha_i}) \quad \text{also written as} \quad M \cong \bigoplus_{\alpha_i} \pi_{\alpha_i}(M) \qquad (0.11)$$

Assuming then, that we have a set of minimal idempotents for both the tube category of the circle and the interval, we can decompose 0.10 in the following manner

$$\mathcal{Z}(\Sigma_1 \cup_I \Sigma_2) \;\cong \bigoplus_{\alpha_i} \pi_{\alpha_i}\big(\mathcal{Z}(\Sigma_1, \{V_{b_i}\}_{b_i \in B_i})\big) \otimes \pi_{\alpha_i}\big(\mathcal{Z}(\Sigma_2, \{V_{b_i}\}_{b_i \in B_i})\big) \tag{0.12}$$

Looking back at the handle decomposition H_Σ and following the line of reasoning that lead us to formula 0.12, we can construct the vector space $\mathcal{Z}(\Sigma)$ by understanding well the Hilbert spaces $\mathcal{Z}(h_k)_{k=0,1,2}$ of each k-handle in the decomposition and the projectors associated to their corresponding attaching boundaries. In particular, for each k-handle we have a Hilbert space

$$\mathcal{H}(h_k) \cong \bigoplus_{\alpha_i :\, \{V_{b_i}\}_{b_i \in B_i} \circlearrowleft} \mathcal{Z}(h_k, \{V_{b_i}\}_{b_i \in B_i}).$$

The degrees of freedom for this space might be split in two: the ones counted by the dimension of $\mathcal{Z}(h_k, \{V_{b_i}\}_{b_i \in B_i})$ which we will call (following Walker's suggestion) "vertex" degrees of freedom and the ones counted by the number of idempotents, α_i, for the tube category in question, and we will call these (also under Walker's suggestion) "idempotent" degrees of freedom.

We can use these local Hilbert spaces, associated to every k-handle, to construct a bigger Hilbert space for the whole surface by tensoring them over all k-handles in the decomposition, for all $k = 0, 1, 2$. To construct a Hamiltoninan we still need a set of commuting projectors. These are provided by the finite set of idempotent projectors π_{α_i}, associated to the tube categories of the attaching boundary of the handles, and bookkeeping projectors that ensure that the idempotents on on both sides of the gluing region actually match. A Hamiltonian can be constructed by summing over all these different types of projectors.

For the reader more familiar with these models this might seem like an awkward method of constructing a Hamiltonian model let alone the Levin-Wen model. In the thesis, we construct the Hamiltonian from a more conventional starting point, a lattice-like structure embedded in a closed surface with "spins" or small Hilbert spaces (relative to the macroscopic Hilbert space of the system) whose Hamiltonian, composed from a finite set of commuting projectors indexed by the plaquettes and vertices of the lattice, has a ground-state naturally isomorphic to the space of string-nets. This mimics the original approach of Levin and Wen [12], which was itself inspired by the seminal work of Kitaev [11] on the Toric Code and is still the most familiar way of describing, working and constructing Hamiltonians for topological phases. Still, we think this approach provides much needed clarity to the conjecture that all fully extended TQFTs can be realized as commuting projector Hamiltonians since it sheds light in the relevancy of finding idempotents in a category and constructing more computable gluing formulas.

In the last section we focus exclusively on the goal of proving that the category of modules of the tube category $\mathrm{Tube}_{\mathcal{C}}(S^1)$ is contravariantly equivalent to the Drinfel'd center of $\mathcal{C}$. This result is typically stated as a folklore result in the "halls" of TQFT and there are a multitude of of paths leading towards its proof, each with its own merits. In Walker's lecture notes [16] (which the author was highly inspired by), a sketch of proof for this result is given by taking the fully-extended data of the String-Net TQFT and comparing the Drinfel'd center of $\mathcal{C}$ with the categorified end of a 2-module for a bicategory with one object. In Kirillov's equally ground-breaking work on Extended Turaev-Viro [10], a proof for this result is proposed although with a different flavour. Kirillov proves that $\mathcal{Z}(\mathcal{C}) \simeq \mathrm{Kar}(\mathrm{Tube}_{\mathcal{C}}(S^1))$, where $\mathrm{Kar}(\mathcal{C})$ is the Karoubi completion (also called Idempotent completion) of $\mathcal{C}$. In particular, his proof omits the balanced structure associated to $\mathrm{Kar}(\mathcal{C})$ and hence fails to establish that the equivalence is braided monoidal. More recently, through the work of Hardiman [8], Hoek [9] and Goosen [7] a fully graphical approach to this proof has emerged that embeds it in the context of fully-extended TQFTs while simultaneously preserving the interpretation of provided by the Karoubi completion suggested by Kirillov. Our version of this proof is really an amalgam of all the other proofs.

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